INTERPOLATION FORMULA FOR THE THICKNESS OF A SHOCK FRONT

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The problem of the thickness of the discontinuity or the structure of a shock wave cannot be solved exactly even in the simplest case of a gas of elastic spheres. This explains the interest being shown in assumptions which simplify the solution of the kinetic equation for a shock wave (as in the well-known Mott-Smith solution). In what follows an attempt is made to restrict the argument to the general principles of statistical mechanics in order to describe the thickness of the shock (without describing its structure quantitatively), making use of simpler, but less likely assumptions. The concrete examples will only be rough illustrations of the proposed method [1] of evaluating the thickness of the shock in various media.

1°. For a certain class of processes, to which shock waves also belong, we postulate, the intuitively clear and fairly general* condition that they are macroscopic in the form of an inequality between the characteristic phase space of the whole macroprocess and the average phase space of the particles of the medium (figure)

$$\Delta x \Delta p_x > \langle \delta x \delta p_x \rangle \approx \frac{h}{2\pi} \exp \left\langle \frac{S}{3Nk} \right\rangle. \quad (1)$$

where the sign $\approx$ is to be understood as the ideal-gas approximation. The states separated by the unknown thickness of the discontinuity Δx (the brackets $\langle \ldots \rangle$ signify an average taken over this thickness), over which the average particle momentum varies by Δp_x, are indistinguishable if this condition is not fulfilled. It is interesting that the well-known property of weak shocks, in which the entropy of the particle S/N varies inappreciably, may be deduced directly from the necessary condition

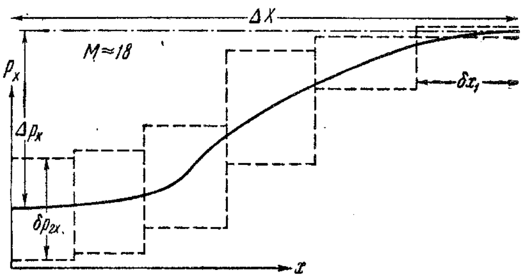
$$\Delta x > \frac{\langle \delta x \delta p_x \rangle}{\Delta p_x} \to \frac{\text{const}}{u_1 - u_2} \to \infty, \qquad \left\{ \begin{matrix} M \to 1 \\ \Delta p_x \equiv m(u_1 - u_2) \to 0 \end{matrix} \right\}.$$

The Rankin-Hugoniat relation for strong shocks gives $\Delta p_x = \delta p_{2x} \sqrt{f}$ (here f is the number of degrees of freedom of a particle (molecule) in the medium, which is assumed to be fixed), i.e., for a shock Mach number $M \to \infty$, inequality (1) reduces to the trivial condition of macroscopy or continuity

$$\Delta x > \langle \delta x \rangle \gtrsim \left\langle \frac{V}{N} \right\rangle^{1/3}. \quad (2)$$

*Generally speaking, the definition of macroscopy is not unique (see [2,3], etc.). Condition (1) is sufficient by reason of the fact that it is invariant relative to canonical transformation of variables appearing in it, as distinct from condition (2), which is trivial.

Here the sign $\gtrsim$ results from the fact that the indeterminancy of the position of a molecule cannot be less that the average distance between molecules.



A typical "instantaneous snapshot" in phase-space μ of the structure of the discontinuity—in the approximation of local thermodynamic equilibrium. The indeterminacy of finding a molecule in phase space is shown by the rectangles. The continuous curve shows the change in the average (hydrodynamic) variables. The shock wave propagates in a gas of elestic spheres ($\Phi = 3.10^{-10}$ m, $(N/V)_1 \approx 10^{27}$ m^{-3}) with the molecular weight of argon. The wave velocity is $\approx$5800 m/sec, the degree of compression is $V_1/V_2 \approx 4$, $T_2/T_1 \approx 100$, $\delta p_{2x} \approx 10 \delta p_{1x} \approx m$ (2500 m/sec), $\delta x_1 \approx 10^{-9}$ m, $\lambda_2 \approx \delta x_2 \approx 6.3 \cdot 10^{-10}$ m.

For a cold medium (when $S/N \to 0$) the general condition for the medium to be macroscopic coincides with Heisenberg's indeterminacy relation

$$\Delta x \Delta p_x > h/2\pi.$$

2°. The simplest relation satisfying condition (1) is a linear one, which is postulated here,

$$\Delta x \Delta p_x \equiv A \langle \delta x \delta p_x \rangle,$$

or

$$\Delta x \equiv A \frac{\langle \delta x \delta p_x \rangle}{\Delta p_x} \approx \frac{A}{\Delta p_x} \frac{h}{2\pi} \exp \left\langle \frac{S}{3Nk} \right\rangle. \quad (3)$$

For central interactions of the molecules the passage of the medium to a new equilibrium state (more exactly, the randomizing of molecule velocities over the three coordinates) requires a minimum of two collisions per molecule on the average, since a single collision can change the particle velocities only in the plane of interaction, i.e., on experiencing one more collision, a molecule (of the type of an elastic sphere) may "forget" its previous distribution (cf. [4]). Whence the minimum "roughness of approximation" of the description of the discontinuity in terms of the variable (Δx, Δp_x),

$$A \equiv \frac{\Delta x \Delta p_x}{\langle \delta x \delta p_x \rangle} \approx \frac{\Delta x}{\langle \delta x \rangle} \frac{\Delta p_x}{\delta p_{2x}}$$

is

$$\lim_{(M\to\infty)} A \approx \frac{2\langle\lambda\rangle \sqrt{f}}{\langle \delta x \rangle} \approx \frac{2\lambda_1}{(V/N)_1^{1/3}} \approx \left\{ \frac{1}{\Phi} \left(\frac{V}{N}\right)_1^{1/3} \right\}^2 \sim \left(\frac{V}{N}\right)_1^{1/3}$$

for a gas of spheres with fixed diameter Φ.

If the indeterminancy of particle momentum [5] in the medium $\delta p_x \equiv \sqrt{mkT}$, then for A independent of the strength of the shock, we obtain the following rough estimate:

$$\Delta x_{\min} \equiv \frac{\lim A}{\Delta p_x} \left\langle \left(\frac{V}{N}\right)^{1/3} (mkT)^{1/2} \right\rangle \approx 2\lambda_1 \frac{\sqrt{fkT_2/m}}{u_1 - u_2} \to 2\lambda_1$$

$$(M \to \infty). \qquad (4)$$

A one-parameter form of this estimate of the minimal shock thickness

$$\Delta x_{\min} \approx \frac{2\langle\lambda\rangle}{M^2 - 1} \sqrt{(M^2 - 1/5)(M^2 + 3)} \to 2\langle\lambda\rangle \quad (M \to \infty) \qquad (5)$$

is obtained* for the more general condition $A \sim \langle\lambda\rangle$ if we use the Rankin-Hugoniot relation and the equation of state of an ideal gas with $f = 3$. In the "equidistributional" approximation formulas similar to (5) are obtained for an arbitrary number f of degrees of freedom of the molecules on the condition that the medium in a one-component gas (see below).

The minimal estimate of the thickness of an isentropic wave (when the equilibrium states are related by Riemann's invariant) should be similarly deduced, and represent the limit of applicability of the isentropic description of the process of increase of the slope of a Riemann wave. Here Liouville's theorem plays a part similar to Heisenberg's indeterminancy relation. When the front of an isentropic wave becomes steep enough, then we may expect, quite independently of the form of the dissipative "mechanisms," the development of fairly strong "phase mixing," which may (for example, in a plasma with a magnetid field) be accompanied by oscillations of the parameters of the medium. Apparently, the establishment of a stationary and equilibrium state lasts appreciably longer than the relaxation time in the discontinuity (cf. [7]). However, we stress that it has not been vigorously demonstrated that real shocks are stationary for any medium.

*This is equivalent to assuming that the Reynolds number relative to the thickness of the shock (cf. [6])

$$\mathrm{Re} \equiv \frac{\Delta u \Delta x}{\langle \nu \rangle} \approx \frac{\Delta u \Delta x}{\langle \delta u \rangle \langle \lambda \rangle} \approx \frac{\Delta u \Delta x}{\langle \delta u \delta x \rangle} \frac{\langle \delta x \rangle}{\langle \lambda \rangle} \approx \frac{\Delta p_x \Delta x}{\langle \delta p_x \delta x \rangle} \frac{1}{A} \equiv 1$$

is independent of the number M of the initial flow in the coordinate system of the shock.

In a "collisionless" plasma counter-beams of ions are formed when a Riemann wave inverts. Judging by certain theoretical ([8], etc.) and experimental [9] data a collisionless shock may arise whose minimal thickness is estimated by the Larmor, Debye or ion characteristic radius r. For such a shock, moving across the magnetic field in an ideal plasma (when one translational degree of freedom of the charged molecule may be disregarded), the estimate is of the form

$$\Delta x_{\min} \approx \frac{C\langle r\rangle}{M^2 - 1} \sqrt{(M^2 - 1/4)(M^2 + 2)} \to C\langle r\rangle \quad (M \to \infty). \qquad (6)$$

Allowing roughly for the fact that the medium has two components

$$C \approx \left(\frac{m_i T_e}{m_e T_i}\right)^{1/2}$$

(here i refers to the ions, e to the electrons), although the effects noted in[10], together with other effects, may restrict the applicability of this formula for C.

The question of the experimental verification of such estimates is as yet open, since the interpretation of the results of astro– and geophysical observations and experiments in shock tubes and other devices is not so simple (it suffices to recall the example [11] of the rejection of previously widely accepted data concerning the structure of the discontinuity in a "collisionless" plasma).

However, verification if possible in the case of a medium comparable to a gas of elastic spheres (i.e., for (4), (5)) which does not manifest appreciable departures from the results of calculations of shock structure carried out by the Monte-Carlo [12] method (computational experiment with molecular spheres). For example, formula (5) gives for the thickness of the shock (for $\langle\lambda\rangle = \lambda_1$):

M	$= 1$	1.5	2	3	∞
$X_{(5)}$	$\approx \infty$	2.625	1.72	1.28	1.00
$X_{[12]}$	≈ 10	2.425	1.53	1.3	?

(here $X \equiv \Delta x / 2\lambda_1$). Similar verifications of the basic assumption (3) compel us to investigate its applicability to other media also, for example, to crystals (then λ is the free path of quasiparticles).

3°. Determination of the range of applicability and a stricter inspection of the consequences of postulate (3) allow us to indicate a class of dissipation mechanisms to which an expansion rule of the type

$$\Delta x = \langle\lambda\rangle F(M), \quad \left\{ \begin{matrix} \infty > F \gtrless 2 \\ 1 \leqslant M \leqslant \infty \end{matrix} \right\} \qquad (7)$$

also extends, as well as other assertions associated with postulate (3).

The tending of the shock thickness of some characteristic length (for example, the mean free path), as the number M increases, is apparently caused by two basic concurrent factors: the macroscopic (hydrodynamic) tendency towards lesser shock thicknesses (the factor F) and the existence of a microscopic mechanism for establishing a new and still more distant* equilibrium state (in (7) the factor $\langle\lambda\rangle$). Thus for dissipation with central interactions between molecules, i. e., forces ~$1/R^S$, the absolute shock thickness may be minimal at moderate M, since the free path

$$\lambda(T, N/V) \sim (T^{2/(s-1)})/(N/V) \to \infty \quad (\infty > s > 2) \qquad \begin{cases} M \to \infty \\ T \to \infty \\ N/V \nrightarrow \infty \end{cases}$$

(cf. [13,14]). Here the roughness of approximation or the "dimensionless number of states" A behaves like the shock thickness

$$A \sim \Delta x \approx \langle\lambda\rangle \to \infty.$$

In particular, for Maxwellian molecules (S = 5), this approximation provides a simple form for the relationship to M (see [15,16]);

$$\lim \Delta x_{\min} \approx \lambda_2 \sim T_2^{1/2} \sim M \to \infty .$$

The greatest gradients of a shock profile with a "thermodynamic" thickness $\lim \Delta x \approx \lambda_1 + \lambda_2$ will clearly be determined by the smallest of the paths, i. e., by the path λ_1 in the case under consideration (cf. [17]).

4°. An interesting consequence of postulate (3) is the prediction of the widening of the shock as a result of a purely nonequilibrium effect, the existence of which does not depend on the dissipation mechanism. This effect may be explained with the help of the following loose discussion. If the average phase volume per unit molecule inside the shock is evaluated (as in the simplified transition from (3) to (4)) according to finite values of $(S/N)_2$ and δp_{2x}, independent of the structure of the shock, then large estimates of the thickness $\Delta x_{\min}$ correspond to large departures from local equilibrium within more intense shocks. Thus the minimal thickness should be obtained in local-equilibrium models of shock front structure.

This assetion is based on the fact that the entropy attains a maximum for an equilibrium Gibbs distribution over a fixed phase region. Thus it is clear that the minimal estimate of the size of the phase volume, i. e., the minimal value of Δx also (for a fixed indeterminancy of momentum), should correspond to a fixed estimate of entropy (as in our case) precisely for a local-equilibrium shock structure. The difference in the estimates of the shock thickness from the Navier-Stokes and kinetic models [17–21] confirms this. Similar general arguments tend to support the assertion that the entropy peak predicted by local-equilibrium models (cf. Fig. 10 in [12a]) decreases as departures from local equilibrium within the shock become larger.

*By definition the shock separates equilibrium states, and so, strictly speaking, $\Delta x = \infty$ for any shock Mach number. However, we are interested in the effective thickness of the shock separating states which are fairly close to the boundary equilibrium states.

5°. For such media as plasmas, which are still problematical in many respects, the accuracy of the estimate of the shock thickness depends on knowing the equilibrium state at least. This is equivalent to knowing the statistical sum or the phase volume of the subsystem under consideration. For transitions from (3) to (4)–(6) we must know the number of degrees of freedom for particles of the medium (in order to solve the problem even in the approximation of equal distribution of energy). Taking into account the electrostatic interaction of particles in a plasma, for example, gives a smaller value of the equilibrium entropy (Debye approximation—see [22]), i. e., a lesser estimate of the thickness of a shock with the same finite temperature. Quasiparticles, among which fluctuations of the electromagnetic field must be counted, may make a substantial contribution to the entropy of the plasma (cf. [23], etc.). We also note that the "hydrodynamic relaxation time" in a plasma may turn out to be appreciably less then the time for establishing local thermodynamic equilibrium [24]. This requires us to refine our concept of the thickness of the shock, and will probably render the interpretation of experimental data more difficult.

An experimental verification of the thickness of the shock is not a trivial matter, not only from the point of view of the ambiguity of the thickness concept [6], but also because of the possible interference in the shock compression process due to the measuring process. For example, a wire-resitance type probe may bring the state within the shock closer to local equilibrium and even "confirm" the theory of shock structure in the Navier-Stokes approximation.

The purest measurements may be expected to result from the application of quantum oscillators working at different wavelengths (in investigations of the shock structure in solids in the gamma- and X-ray range).

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